14-6 Double-Angle and Half-Angle Formulas

Main Ideas

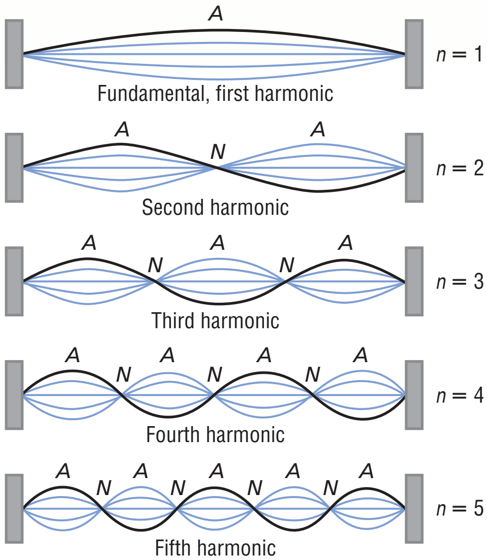
- Find values of sine and cosine involving double-angle formulas.
- Find values of sine and cosine involving half-angle formulas.

New Vocabulary

double-angle formulas
half-angle formula

GET READY for the Lesson

Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called *nodes*, that do not move under the vibration. Halfway between each pair of consecutive nodes are *antinodes* that undergo the maximum vibration. The nodes and antinodes form *harmonics*. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2}\theta$.



Double-Angle Formulas You can use the formula for $\sin(\alpha + \beta)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the formula for $\cos(\alpha + \beta)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta\end{aligned} \qquad \begin{aligned}\cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta. \\ &= 1 - 2\sin^2\theta && \text{Simplify.} \\ \cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta. \\ &= 2\cos^2\theta - 1 && \text{Simplify.}\end{aligned}$$

These formulas are called the **double-angle formulas**.

KEY CONCEPT — *Double-Angle Formulas*

The following identities hold true for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \cos 2\theta &= 1 - 2\sin^2\theta \\ \cos 2\theta &= 2\sin^2\theta - 1\end{aligned}$$

EXAMPLE 1 Double-Angle Formulas

Find the exact value of each expression if $\sin\theta = \frac{4}{5}$ and θ is between 90° and 180°.

a. $\sin 2\theta$

Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

First, find the value of $\cos\theta$.

$\cos^2\theta = 1 - \sin^2\theta$ — $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2$ — $\sin\theta = \frac{4}{5}$

$\cos^2\theta = \frac{9}{25}$ — Subtract.

$\cos\theta = \pm\frac{3}{5}$ — Find the square root of each side.

Since θ is in the second quadrant, cosine is negative. Thus, $\cos\theta = -\frac{3}{5}$.

Now find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ — Double-angle formula

$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ — $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$

$= -\frac{24}{25}$ — Multiply.

The value of $\sin 2\theta$ is $-\frac{24}{25}$.

b. $\sin 2\theta$

Use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$\cos 2\theta = 1 - 2\sin^2\theta$ — Double-angle formula

$= 1 - 2\left(\frac{4}{5}\right)^2$ — $\sin\theta = \frac{4}{5}$

$= -\frac{7}{25}$ — Simplify.

The value of $\cos 2\theta$ is $-\frac{7}{25}$.

CHECK Your Progress

Find the exact value of each expression if $\cos = -\frac{1}{3}$ and $90° < \theta < 180°$.

1A. $\sin 2\theta$

1B. $\cos 2\theta$

Online Personal Tutor at algebra2.com

Half-Angle Formulas You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find $\sin\frac{\alpha}{2}$.

$1 - 2\sin^2\theta = \cos 2\theta$ — Double-angle formula

$1 - 2\sin^2\frac{\alpha}{2} = \cos\alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$ — Solve for $\sin^2\frac{\alpha}{2}$.

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ — Take the square root of each side.

Find $\cos \frac{\alpha}{2}$.

$2\cos^2 \theta - 1 = \cos 2\theta$ — Double-angle formula

$2\cos^2 \frac{\alpha}{2} - 1 = \cos \alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}$ — Solve for $\cos^2 \frac{\alpha}{2}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ — Take the square root of each side.

These are called the **half-angle formulas**. The signs are determined by the function of $\frac{\alpha}{2}$.

KEY CONCEPT — ***Half-Angle Formulas***

The following identities hold true for all values of α.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ $\qquad$ $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$

EXAMPLE Half-Angle Formulas

2 **Find $\cos \frac{\alpha}{2}$ if $\sin \alpha = -\frac{3}{4}$ and α is in the third quadrant.**

Since $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$, we must find $\cos \alpha$ first.

$\cos^2 \alpha = 1 - \sin^2 \alpha$ — $\cos^2 \alpha + \sin^2 \alpha = 1$

$\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2$ — $\sin \alpha = -\frac{3}{4}$

$\cos^2 \alpha = \frac{7}{16}$ — Simplify.

$\cos \alpha = \pm\frac{\sqrt{7}}{4}$ — Take the square root of each side.

Since α is in the third quadrant, $\cos \alpha = \frac{\sqrt{7}}{4}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ — Half-angle formula

$= \pm\sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{2}}$ — $\cos \alpha = -\frac{\sqrt{7}}{4}$

$= \pm\sqrt{\frac{4 - \sqrt{7}}{8}}$ — Simplify the radicand.

$= \pm\frac{\sqrt{4 - \sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ — Rationalize.

$= \pm\frac{\sqrt{8 - 2\sqrt{7}}}{4}$ — Multiply.

Since α is between 180° and 270°, $\frac{\alpha}{2}$ is between 90° and 135°. Thus, $\cos \frac{\alpha}{2}$ is negative and equals $-\frac{\sqrt{8 - 2\sqrt{7}}}{4}$.

Study Tip

Choosing the Sign

You may want to determine the quadrant in which the terminal side of $\frac{\alpha}{2}$ will lie in the first step of the solution. Then you can use the correct sign from the beginning.

CHECK Your Progress

2. Find $\sin \frac{\alpha}{2}$ if $\sin \alpha = \frac{2}{3}$ and α is in the 2nd quadrant.

EXAMPLE 3 Evaluate Using Half-Angle Formulas

Find the exact value of each expression by using the half-angle formulas.

a. sin 105°

$$\sin 105° = \sin \frac{210°}{2}$$

$$= \sqrt{\frac{1 - \cos 210°}{2}} \qquad \sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$

$$= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}} \qquad \cos 210° = -\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{2 + \sqrt{3}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2 + \sqrt{3}}}{2} \qquad \text{Simplify the denominator.}$$

b. $\cos \frac{\pi}{8}$

$$\cos \frac{\pi}{8} = \frac{\frac{\pi}{4}}{2}$$

$$= \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} \qquad \cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

$$= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \qquad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$= \sqrt{\frac{2 + \sqrt{2}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2 + \sqrt{2}}}{2} \qquad \text{Simplify the denominator.}$$

CHECK Your Progress

3A. sin 135° **3B.** $\cos \frac{7\pi}{8}$

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

EXAMPLE 4 Verify Identities

Verify that $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$ is an identity.

$$(\sin \theta + \cos \theta)^2 \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Original equation}$$

$$\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Multiply.}$$

$$1 + 2 \sin \theta \cos \theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \sin 2\theta = 1 + \sin 2\theta \qquad \text{Double-angle formula}$$

CHECK Your Progress

4. Verify that $4 \cos^2 x - \sin^2 2x = 4 \cos^4 x$

CHECK Your Understanding

Examples 1, 2 (pp. 854–855)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

1. $\cos \theta = \frac{3}{5}$; $0° < \theta < 90°$

2. $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

3. $\sin \theta = \frac{1}{2}$; $0° < \theta < 90°$

4. $\sin \theta = -\frac{3}{4}$; $270° < \theta < 360°$

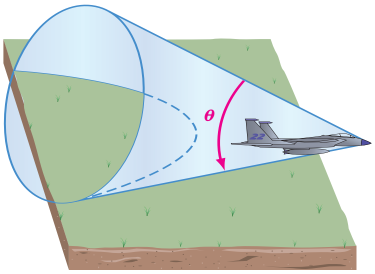
Example 3 (p. 856)

Find the exact value of each expression by using the half-angle formulas.

5. $\sin 195°$

6. $\cos \frac{19\pi}{12}$

7. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If θ is the measure of the angle at the vertex of the cone, then the Mach number M can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°.



Example 4 (p. 856)

Verify that each of the following is an identity.

8. $\cot x = \frac{\sin 2x}{1 - \cos 2x}$

9. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Exercises

HOMEWORK	
For Exercises	**See Examples**
10–15	1, 2
16–21	3
22–27	4

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

10. $\sin \theta = \frac{5}{13}$; $90° < \theta < 180°$

11. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

12. $\cos \theta = -\frac{1}{3}$; $180° < \theta < 270°$

13. $\sin \theta = -\frac{3}{5}$; $180° < \theta < 270°$

14. $\sin \theta = -\frac{3}{8}$; $270° < \theta < 360°$

15. $\cos \theta = -\frac{1}{4}$; $90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

16. $\cos 165°$

17. $\sin 22\frac{1}{2}°$

18. $\cos 157\frac{1}{2}°$

19. $\sin 345°$

20. $\sin \frac{7\pi}{8}$

21. $\cos \frac{7\pi}{12}$

Verify that each of the following is an identity.

22. $\sin 2x = 2 \cot x \sin^2 x$

23. $2 \cos^2 \frac{x}{2} = 1 + \cos x$

24. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

25. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

26. $\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$

27. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$

PHYSICS **For Exercises 28 and 29, use the following information.**

An object is propelled from ground level with an initial velocity of v at an angle of elevation θ.

28. The horizontal distance d it will travel can be determined using the formula $d = \frac{v^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan \theta - \tan \theta \sin^2 \theta)$.

29. The maximum height h the object will reach can be determined using the formula $d = \frac{v^2 \sin^2 \theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled.

Find the exact values of sin 2θ, cos 2θ, sin $\frac{\theta}{2}$, and cos $\frac{\theta}{2}$ for each of the following.

30. $\cos \theta = \frac{1}{6}$; $0° < \theta < 90°$

31. $\cos \theta = -\frac{12}{13}$; $180° < \theta < 270°$

32. $\sin \theta = -\frac{1}{3}$; $270° < \theta < 360°$

33. $\sin \theta = -\frac{1}{4}$; $180° < \theta < 270°$

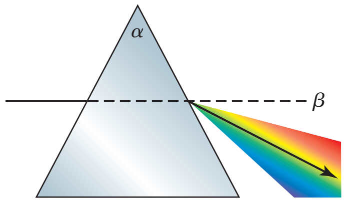
34. $\cos \theta = \frac{2}{3}$; $0° < \theta < 90°$

35. $\sin \theta = \frac{2}{5}$; $90° < \theta < 180°$

Real-World Link

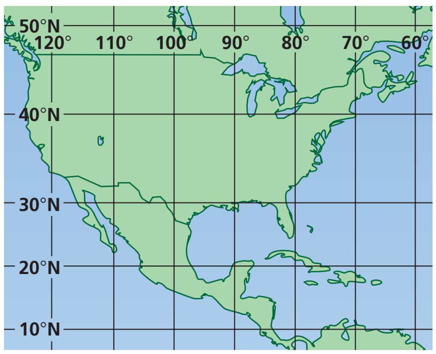
A rainbow appears when the sun shines through water droplets that act as a prism.

36. **OPTICS** If a glass prism has an apex angle of measure α and an angle of deviation of measure β, then the index of refraction n of the prism is given by $n = \frac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\alpha}{2}}$. What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2?



GEOGRAPHY **For Exercises 37 and 38, use the following information.**

A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.



37. Write this expression in terms of a trigonometric function of L.

38. Find the exact value of the expression if $L = 60°$.

EXTRA PRACTICE
See pages 924, 939.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. **REASONING** Explain how to find $\cos \frac{x}{2}$ if x is in the third quadrant.

40. **REASONING** Describe the conditions under which you would use each of the three identities for $\cos 2\theta$.

41. **OPEN ENDED** Find a counterexample to show that $\cos 2\theta = 2 \cos \theta$ is not an identity.

42. ***Writing in Math*** Use the information on page 853 to explain how trigonometric functions can be used to describe music. Include a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next and an explanation of what happens to the period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic.

STANDARDIZED TEST PRACTICE

43. ACT/SAT Find the exact value of $\cos 2\theta$ if $\sin \theta = \frac{-\sqrt{5}}{3}$ and $180° < \theta < 270°$.

A $\frac{-\sqrt{6}}{6}$

B $\frac{-\sqrt{30}}{6}$

C $\frac{-4\sqrt{5}}{9}$

D $\frac{-1}{9}$

44. REVIEW Which of the following is equivalent to $\frac{\cos \theta\,(\cot^2 \theta + 1)}{\csc \theta}$?

F $\tan \theta$

G $\cot \theta$

H $\sec \theta$

J $\csc \theta$

Spiral Review

Find the exact value of each expression. (Lesson 14-5)

45. $\cos 15°$ **46.** $\sin 15°$ **47.** $\sin (-135°)$

48. $\cos 150°$ **49.** $\sin 105°$ **50.** $\cos (-300°)$

Verify that each of the following is an identity. (Lesson 14-4)

51. $\cot^2 \theta - \sin^2 \theta = \frac{\cos^2 \theta \csc^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta}$

52. $\cos \theta\,(\cos \theta + \cot \theta) = \cot \theta \cos \theta\,(\sin \theta + 1)$

ANALYZE TABLES **For Exercises 53 and 54, use the following information.**

The magnitude of an earthquake M measured on the Richter scale is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. (Lesson 9-2)

53. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?

54. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock?

Strongest Earthquakes in 20th Century

Location, Year	Magnitude
Chile, 1960	9.5
Alaska, 1964	9.2
Russia, 1952	9.0
Ecuador, 1906	8.8
Alaska, 1957	8.8
Kuril Islands, 1958	8.7
Alaska, 1965	8.7
India, 1950	8.6
Chile, 1922	8.5
Indonesia, 1938	8.5

Source: U.S. Geological Survey

Write each expression in quadratic form, if possible. (Lesson 6-6)

55. $a^8 - 7a^4 + 13$ **56.** $5n^7 + 3n - 3$ **57.** $d^6 + 2d^3 + 10$

Find each value if $f(x) = x^2 - 7x + 5$. (Lesson 2-1)

58. $f(2)$ **59.** $f(0)$ **60.** $f(-3)$ **61.** $f(n)$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 5-3)

62. $(x + 6)(x - 5) = 0$ **63.** $(x - 1)(x + 1) = 0$

64. $x(x + 2) = 0$ **65.** $(2x - 5)(x + 2) = 0$

66. $(2x + 1)(2x - 1) = 0$ **67.** $x^2(2x + 1) = 0$

EXPLORE 14-7

Graphing Calculating Lab

Solving Trigonometric Equations

The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83/84 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

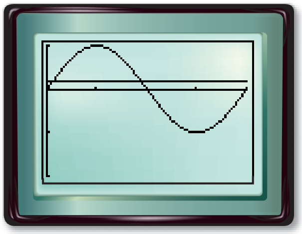
ACTIVITY 1

Use a graphing calculator to solve $\sin x = 0.2$ if $0° \leq x < 360°$.

Rewrite the equation as two functions, $y = \sin x$ and $y = 0.2$. Then graph the two functions. Look for the point of intersection.

Make sure that your calculator is in degree mode to get the correct viewing window.

KEYSTROKES: [MODE] [▼] [▼] [▶] [ENTER] [WINDOW] 0 [ENTER] 360 [ENTER] 90 [ENTER] −2 [ENTER] 1 [ENTER] 1 [ENTER] [Y=] [SIN] [X,T,θ,n] [ENTER] 0.2 [ENTER] [GRAPH]



[0, 360] scl: 90 by [−2, 1] scl: 1

Based on the graph, you can see that there are two points of intersection in the interval $0° \leq x < 360°$. Use [ZOOM] or [2nd] [CALC] 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

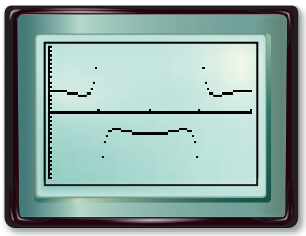
Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

ACTIVITY 2

Use a graphing calculator to solve $\tan^2 x \cos x + 5 \cos x = 0$ if $0° \leq x < 360°$.

Because the tangent function is not continuous, place the calculator in Dot mode. The related functions to be graphed are $y = \tan^2 x \cos x + 5 \cos x$ and $y = 0$.

These two functions do not intersect. Therefore, the equation $\tan^2 x \cos x + 5 \cos x = 0$ has no real solutions.



[0, 360] scl: 90 by [−15, 15] scl: 1

EXERCISES

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.8$ if $0° \leq x < 360°$

2. $\tan x = \sin x$ if $0° \leq x < 360°$

3. $2 \cos x + 3 = 0$ if $0° \leq x < 360°$

4. $0.5 \cos x = 1.4$ if $-720° \leq x < 720°$

5. $\sin 2x = \sin x$ if $0° \leq x < 360°$

6. $\sin 2x - 3 \sin x = 0$ if $-360° \leq x < 360°$

Other Calculator Keystrokes at algebra2.com